A Vintage Model to Production and Consumption
by Manfred Jäger and Gunter Steinmann

Martin-Luther-Universität Halle-Wittenberg
Wirtschaftswissenschaftliche Fakultät

www.wiwi.uni-halle.de
A Vintage Model to Production and Consumption

Manfred Jäger, Gunter Steinmann

Martin-Luther-University Halle-Wittenberg
Department of Economics (Wiwi)
D-06099 Halle (Germany)
Email: jaeger@wiwi.uni-halle.de (Manfred Jäger), steinmann@wiwi.uni-halle.de (Gunter Steinmann)
Fax: 0049/345/5527188
Tel: 0049/345/5523320

Abstract

This paper applies the vintage approach to production and consumption. We postulate different vintages of consumer goods and presume a specific form of habit persistence of consumers. Consumers buy the goods and services they became acquainted with during childhood and adolescence and resist to purchase more modern goods and services innovated in later stages of their lives. The preference of older age groups in favor of well known goods and services can be explained by the fact that the purchase of new goods and services is not sufficient to be able to consume them.

The major result of the paper is that a non-neoclassical link between population growth and economic welfare can occur. A lower population growth rate may imply a lower consumption level. This happens if the adjustment of the vintage structure of production to the age structure of the population is unfavorable in terms of the relative productivity of the capital-vintages.

Key words: Vintage Model, Economic growth, Aging, Population Growth
JEL classification: O41, J10

1 Introduction

The objective of this paper is to analyze the influence of the age structure of population on the growth and structure of production and consumption. We specifically argue that the positive effects of a population decline must be weighted against its negative effects. A declining population alleviates the constraints from capital dilution. But the "skewed" age structure coherent to a population decline adds the new hardship of reshaping the production structure. Aging leads to decreasing investment in new capital and to a lower share of new goods and services in output. Our analysis shows that lower population growth and population decline reduces the steady state consumption level of pensioners for reasonable values of the parameters.

The vintage concept is applicable both for production functions and for consumption functions. The assumption of heterogeneous vintages of inputs instead of homogeneous
labor and capital is the appropriate hypothesis to analyze of the production process, since the newest technical knowledge is fully embodied in the newest physical and human capital but is absent in the physical and human capital invested in previous periods (Solow (1959), Chari and Hopenhayn (1993) and Iacopetto (2002) for a survey).

The same argument holds to the analysis of consumption too. The households buy goods and services and utilize them as inputs at their home production (Becker, 1965). The home production of households depends upon their technical know-how and this links technical knowledge with consumption. We assume that the technical knowledge of consumers varies with their age. Adolescents and young adults acquire the newest technical knowledge more easily. And in addition, according to the theory of human capital, the incentive to invest in new knowledge is stronger for younger than for older generations. It is a well-known fact that teenagers are open to buy new commodities and to try a new way of life. They consume new goods and services that their parents are not willing or not able to consume. The latter often maintain throughout their lives the preferences formed in the years of their youth. The thesis that the consumption pattern is shaped in the days of childhood and youth makes the vintage approach attractive for the analysis of consumption. To our knowledge, the approach of this paper is new to the literature.

We employ this idea to analyze the consequences of a change in the age structure of the population. In second section we develop the model. In section 3 we discuss the link between age structure, production structure and consumption. The fourth section considers the case of an endogenous saving rate. Section 5 concludes.

2 The Model with a Constant Saving Rate

2.1 The Assumptions

We implement the vintage approach both in the production and in the consumption functions of our model. For the sake of simplicity we assume Cobb Douglas functions with homogenous labor and confine the vintage hypothesis on capital. The firms produce two kinds of goods in each period $t$:

- A new good of novel quality unknown before period $t$. The new good is produced with the most modern capital vintage $K_t(t)$ and labor $L_t(t)$ of standard quality (since labor is homogeneous in our model, the subscript refers to the capital vintage the labourers work with)

\[ X_t(t) = K_t(t)^a L_t(t)^{1-a}. \]  

- A mature good of well known quality that has been invented in the previous period. The mature good is produced with capital of vintage $t - 1$ and labour of standard
More mature goods of quality \( t - \tau, \tau > 1 \) invented prior to period \( t - 1 \) are neither produced nor consumed any more, i.e. we restrict our analysis to two goods: the innovative good of period \( t \) and the mature good of period \( t - 1 \). As consequence of this assumption all older capital vintages \( t - \tau, \tau > 1 \) are worthless and have been fully depreciated.

We assume three overlapping generations: the dependent generation of age 0 ("the children") \( N_0(t) \), the active generation of age 1 ("the parents") \( N_1(t) \) and the inactive generation of age 2 ("the grandparents") \( N_2(t) \). The children and grandparents are solely consumers while the parents are consumers and workers. We consider labour as homogeneous since labour is exclusively supplied by the active generation. The labourers produce the new good \( t \) with capital vintage \( K_t(t) \) and the mature good \( t - 1 \) with the capital vintage \( K_{t-1}(t) \). We assume full employment:

\[
N_1(t) = L_t(t) + L_{t-1}(t).
\]

(3)

Since the members of age group 1 ("the parents") have finished their school education more recently and gain further experience on the job, their technical know-how will be sufficient to use most innovations. We suppose that members of the active age are both willing and able to consume the new good of quality \( t \). Children are dependent upon the decisions of their parents. Their consumption can be thought as part of the parental consumption. The consumption expenditures of children and parents are therefore

\[
P_t(t)C_t(t) = (1 - s)Y_1(t),
\]

(4)

where \( C_t(t) \) denotes consumption demand for the new good of quality \( t \) by the generations of parents and children, \( P_t(t) \) the price of the new good of quality \( t \), \( s \) the saving rate and \( Y_1(t) \) the total income earned by parents (age group 1).

The members of age group 2 ("the grandparents") are assumed to consume the mature good of quality \( t - 1 \) that they got to know and to use in previous period \( t - 1 \) when they belonged to the active age group and when the good was introduced into the market. This hypothesis corresponds to the observation that older persons primarily buy conventional goods they are acquainted with but that they hesitate and resist to purchase new goods and services they are unfamiliar with. They lack the know-how to handle the new goods and do not have use for them at their home production.

The pensioners finance their consumption expenditures from two sources. (1) They earn income by renting capital to firms and (2) they receive a revenue of \( V_t(t)K_t(t) \) from selling capital property to the active generation (\( V_t(t) \) is the market price of capital of vintage \( t \)). Notice, that the property on capital vintage \( K_{t-1}(t) \) becomes worthless after period \( t \). We assume that the pensioners do not save. They spend their interest income
and their proceeds fully on consumption:

\[ C_{t-1}(t)P_{t-1}(t) = R_t(t)K_t(t) + V_t(t)K_t(t) + R_{t-1}(t)K_{t-1}(t), \]  

(5)

where \( R_t(t) \) denotes the rental rate on capital of vintage \( t \).

The differences between output and (real) consumption give the investment in the new good \( t \) and the investment in the mature good \( t-1 \).

\[ I_t(t) = X_t(t) - C_t(t) \geq 0 \]  

(6)

\[ I_{t-1}(t) = X_{t-1}(t) - C_{t-1}(t) \geq 0. \]  

(7)

The producers of capital equipment manufacture the investment \( t \) and the investment \( t-1 \) into the new capital vintage \( t+1 \). The new capital vintage will be utilized in period \( t+1 \). It allows the innovation and production of a new good of novel quality \( t+1 \). The manufacturing of capital equipment is described by the Cobb Douglas function

\[ K_{t+1}(t) = I_t(t)^\sigma I_{t-1}(t)^{1-\sigma}. \]  

(8)

The members of the active age group buy the new capital (new issues) \( K_{t+1}(t) \) as well as the older capital (blue chips) \( K_t(t) \) on sale by the pensioners. Notice, that capital vintage \( K_{t-1}(t) \) is not demanded since the production of the good \( t-1 \) ends with period \( t \). The disappearance of good \( t-1 \) in period \( t+1 \) makes the capital equipment of vintage \( t-1 \) obsolete and worthless at the end of period \( t \).

The active generation finances their purchases of capital with their savings.

\[ K_{t+1}(t)V_{t+1}(t) + K_t(t)V_t(t) = sY_1(t) \]  

(9)

We assume that individuals neither buy nor inherit property before they enter into their active age. Therefore the pensioners earn the capital income and the active persons receive the wage income

\[ Y_1(t) = W(t)N_1(t). \]  

(10)

Perfect competition on all markets implies that the real returns to the factors of production are equal to their respective marginal products:

\[ \frac{W(t)}{P_t(t)} = \frac{\partial X_t(t)}{\partial L_t}, \quad \frac{W(t)}{P_{t-1}(t)} = \frac{\partial X_{t-1}(t)}{\partial L_{t-1}} \]  

(11)

\[ \frac{R_t(t)}{P_t(t)} = \frac{\partial X_t(t)}{\partial K_t}, \quad \frac{R_{t-1}(t)}{P_{t-1}(t)} = \frac{\partial X_{t-1}(t)}{\partial K_{t-1}} \]  

(12)

\[ \frac{P_t(t)}{V_{t+1}(t)} = \frac{\partial K_{t+1}(t)}{\partial I_t}, \quad \frac{P_{t-1}(t)}{V_{t+1}(t)} = \frac{\partial K_{t+1}(t)}{\partial I_{t-1}} \]  

(13)

In period \( t \) the households can buy one unit of capital \( K_{t+1}(t) \) \([K_t(t)]\) at the price of \( V_{t+1}(t) \) \([V_t(t)]\). They expect to earn an interest income and in addition to get money
from the sale of their property at the end of the period

\[ R^e_{t+1}(t + 1) + V^{e}_{t+1}(t + 1), \]
\[ R^e_t(t + 1) + V^e_t(t + 1), \]

where \( R^e_{t+1}(t + 1) \) is the expected interest income from capital vintages \( t + 1 \) and \( V^e_{t+1}(t + 1) \) is expected sales price of capital vintages \( t+1(t) \) in period \( t+1 \) (the superscripts \( e \) indicate expected values). The capital of vintage \( t \) will be worthless at the end of period \( t+1 \) because it will be out of use after that period:

\[ V^e_t(t + 1) = 0 \]

Since both capital vintages must yield the same rate of return, we obtain the following arbitrage condition

\[ \frac{R^e_{t+1}(t + 1) + V^{e}_{t+1}(t + 1)}{V^e_{t+1}(t)} = \frac{R^e_t(t + 1)}{V^e_t(t)}. \] (14)

We add the well known equations describing the population dynamics and the age structure of stable populations:

\[ N_1(t) = (1 + n)N_1(t - 1), \] (15)
\[ N_0(t) = \frac{1 + n}{1 - m_0}N_1(t), \] (16)
\[ N_2(t) = \frac{1 - m_1}{1 + n}N_1(t). \] (17)

The growth rate of population is constant and the age structure of the stable population is determined by the growth rate of population and by the mortality rates \( m_0, m_1 \) for the two age groups 0 and 1. For sake of simplicity, we assume \( m_0 = m_1 = 0 \). The age structure of the population is solely determined by the growth rate of population, e.g. the ratio of worker and pensioners is given by \( 1 + n \).

We take the price of capital vintage \( t \) as numeraire

\[ V_t(t) = 1 \] (18)

The last missing elements of the model concern the assumptions on expectation formation. We assume perfect foresight

\[ R^e_{t+1}(t + 1) = R_{t+1}(t + 1), \] (19)
\[ R^e_t(t + 1) = R_t(t + 1), \] (20)
\[ V^e_{t+1}(t + 1) = V_{t+1}(t + 1). \] (21)

Since the model is deterministic the assumption of perfect foresight corresponds to rational expectations in our framework.
2.2 Temporary Equilibrium and the Dynamics of Capital Accumulation

To characterize the temporary equilibrium we define the variable $\gamma_V$ as ratio of the prices of capital vintages $t+1$ and $t$:

$$\gamma_V(t+1) := \frac{V_{t+1}(t)}{V_t(t)} = \frac{R_{t+1}(t+1) + V_{t+1}(t+1)}{R_t(t+1)} \quad (22)$$

where the second equation is a restatement of the arbitrage condition (14). Note that $V_{t+1} = \gamma_V V_t = \gamma_V$.

**Proposition:** *In the temporary equilibrium we have*

$$\gamma_V(t+1) = \frac{W(t)}{\Theta \left( \frac{K_t(t)}{L_t(t)} \right)^\alpha \left( \frac{K_{t-1}(t)}{K_t(t)} \right)^\alpha \left( \frac{L_t(t)}{N_0(t)} \right)^\alpha (1-\sigma)} \quad (23)$$

where

$$W(t) = \frac{K_t(t)\sigma(1-\alpha)}{sN_1(t)\sigma(1-\alpha) - L_t(t) + (1-s)(1-\alpha)N_1(t)}, \quad (24)$$

$$\Theta = \sigma(1-\alpha)\left(\frac{1-\sigma}{\sigma}\right)^{(1-\sigma)}.$$

Equations (24) and (23) describe the temporary equilibrium of period $t$. The only current endogenous variable is $L_t(t)$. The forward looking variable $\gamma_V(t+1)$, the capital vintages $K_t(t)$, $K_{t-1}(t)$, and the population profile $N_0(t), N_1(t), N_2(t)$ determine the dynamics of capital accumulation since the temporary equilibrium of period $t$ fixes $K_{t+1}(t+1)$. Thus we can write the dynamic system for the capital vector as follows:

$$\begin{pmatrix} K_{t+1}(t+1) \\ K_t(t+1) \end{pmatrix} = \Phi(\gamma_V(t+1), K_t(t), K_{t-1}(t), N_0(t), N_1(t), N_2(t)) \begin{pmatrix} K_t(t) \end{pmatrix}$$

for some function $\Phi$.

2.3 Steady State Condition

Since there is no technical progress and the mechanisms of capital accumulation resembles those of the Solow growth model, the equilibrium growth rate of capital is equal to the population growth rate $n$. Therefore, in the steady state $K_{t+1}(t) = (1+n)K_t(t) = \gamma_N K_t(t)$ holds. In the appendix we proof the following characterization of the steady state.

**Proposition:** *The equilibrium growth path is completely characterized by the following equation for the capital intensity $k_t(t) = K_t(t)/L_t(t)$*

$$k_t = \left( \frac{\Theta \gamma_V \gamma_N^{\alpha(\sigma-1)}}{\Gamma(\frac{1+\gamma_N \gamma_V}{\gamma_N - 1})^{\alpha(1-\sigma)}} \right)^{\frac{1}{\gamma}} \quad (25)$$

\[\text{See the appendix for a proof that there is a temporary equilibrium for all } \gamma_V(t+1).\]
where
\[
\Gamma = \frac{\chi (1 + \gamma_N \gamma_V)}{s} - \sigma (1 - \alpha) \quad (26)
\]
\[
\chi = (s \sigma (1 - \alpha) + (1 - s)(1 - \alpha)) \quad (27)
\]
are auxiliary variables and the condition
\[
\gamma_V \gamma_N = \frac{s (1 + \alpha \Gamma - \alpha)}{\alpha (1 + \gamma_N \gamma_V - s \Gamma)} \quad (28)
\]
implicitly defines the steady state value of \( \gamma_V \).

Note that this equation determines the product \( \gamma_V \gamma_N \) in terms of exogenous parameters. Thus an increase of \( \gamma_N \) by one percent leads c.p. to a one percent decline of \( \gamma_V \), i.a.W the elasticity of \( \gamma_V \) with respect to \( \gamma_N \) is \(-1\). We will repeatedly use the fact that the product \( \gamma_N \gamma_V \) does not depend on \( \gamma_N \). Consider a decline of the population growth rate and consequently a more aged population (with a relatively high share of the not-active age group (“grandparents”)). The aging of population increases the demand for mature goods and services and enforces a process of restructuring. Resources must be diverted from the production of new goods and services to the production of mature goods and services. The restructuring requires a reallocation of capital (and labor) from the innovative sector to the mature sector. In equilibrium the fraction of new issues relative to blue ships is lower. Individuals hold less new issues if these are relatively expensive, i.e. if \( \gamma_V \) is relatively high.

The long run values of all other endogenous variables can be calculated using this equilibrium conditions, e.g.
\[
W(t) = \Gamma k(t), \quad (29)
\]
\[
\frac{N_i(t)}{K_i(t)} = \frac{1 + \gamma_N \gamma_V}{s W(t)}. \quad (30)
\]

3 Age Structure, Production Structure and Consumption Levels

What are the effects of the population growth rate \( n \) on the steady state production structure \( X_i(t)/X_{i-1}(t) \) of the economy and on the per capita consumption levels of the parents \( c_i(t) \) and the grandparents \( c_{i-1}(t) ? \) According to the Solow model lower population growth increases the steady state capital intensity and per capita consumption. In our framework the link between population growth and equilibrium consumption is more complex since population growth affects the age structure of the population and thereby the production structure of the economy.

\( ^3 \)Note, that \( \Gamma \) depends on the product \( \gamma_V \gamma_N \).
With a stable and stationary population $\gamma_N = 1$ the ratio $N_1/N_2$ between individuals of age group 1 and individuals of age group 2 equals 1. With a stable and declining population $\gamma_N < 1$ and this ratio gets smaller than 1, viz. $\gamma_N$. Compared with a growing population a declining population has relatively more pensioners and c.p. less demand for the innovative good. Hence, lower population growth is related to a production structure with less innovative goods. Growth theorists ignore the vintage structure of consumption and neglect this effects of the age structure of the population. This neglect is a severe failure of traditional growth theory and mainstream thinking about the role of population growth. Taking into account the vintage structure of consumption changes the assessment of a lower population growth rate. We will show that lower population growth always increases the steady state level of consumption of the middle generation $c_t$ but the effect on the steady state level of consumption $c_{t-1}$ of the older generation is ambiguous.

A smaller $\gamma_N$ can lead to lower or to higher consumption of the "grandparents" depending on the supply-side parameters of the economy, i.e. on the parameters $\alpha$ and $\sigma$ of the production functions.

The steady state conditions (29) and (30) allow us to calculate the equilibrium fraction of workers employed in the first sector

$$\frac{L_t(t)}{N_1(t)} = \frac{s \Gamma}{1 + \gamma_N \gamma_V}.$$ 

This fraction does not depend on $\gamma_N$ since $\Gamma$ is independent of $\gamma_N$ and the product $\gamma_V \gamma_N$ is constant. We can use this equation to derive the steady structure of production:

$$\frac{X_t(t)}{X_{t-1}(t)} = \left( \frac{K_t(t)}{K_{t-1}(t)} \right)^{\alpha} \left( \frac{L_t(t)}{L_{t-1}(t)} \right)^{1-\alpha} = \gamma_N^\alpha \left( \frac{N_1(t)}{L_t(t)} - 1 \right)^{\alpha-1} = \gamma_N^\alpha \left( \frac{1 + \gamma_N \gamma_V - 1}{s \Gamma} \right)^{\alpha-1} = \gamma_N^\alpha.$$

$\gamma_N$ only affects the production structure via the first factor $\gamma_N^\alpha$ since $\gamma_N \gamma_V$ is constant. Therefore, the sectoral adjustment to demographic change is completely driven by capital adjustment. The necessary adjustment of the production structure is difficult to accomplish if $\alpha$ is small. Consider

$$\frac{X_t(t)}{X_{t-1}(t)} = \left( \frac{K_t(t)}{K_{t-1}(t)} \right)^{\alpha} \left( \text{const.} \right)^{1-\alpha}.$$

An adjustment of the production, i.e. the shift in the ratio $X_t(t)/X_{t-1}(t)$ by 1 percent requires an adjustment of the capital structure $K_t(t)/K_{t-1}(t)$ by $\frac{1}{\alpha}$ percent. This means that for small values of $\alpha$ an economy needs relative large changes of the capital structure to reach the necessary adjustment of production.

The elasticity of the production structure $X_t/X_{t-1}$ with respect to $\gamma_N$ is given by:

$$\frac{\partial (X_t/X_{t-1})}{\partial \gamma_N} \frac{\gamma_N}{(X_t/X_{t-1})} = \alpha.$$
Thus, if the population growth rate declines by one percent and the population correspondingly becomes older, then the production structure shifts by $\alpha$ per cent towards the production of the mature goods. This leads to a “gap” between the shifts in the demand and supply structures. In our model demand is mainly determined by the age structure of the population. Thus, whereas demand changes by one percent supply is only incompletely adjusted by $\alpha$ percent. The gap is responsible for the main result of this section to be discussed below.

A declining and aging population does not only affect the structure of production but also influences the shape of consumption. First, we consider the effect of the rate of population growth on per capita consumption of the active generation (the “parents”). We have (see equations (10) and (11))

$$c_t = \frac{C_t(t)}{N_1(t)} = (1-s)(1-\alpha)k_t^\alpha.$$ (31)

A decrease of the rate of population growth reduces the negative effect of capital dilution and increases the capital intensity $k_t$ in the innovative sector. The negative relation between $k_t$ and $\gamma_N$ can be derived from equation (25) (note that the denominator of the right side of (25) is independent of $\gamma_N$). The high capital labor ratio in the modern sector stimulates the labor productivity and reduces the prices of the innovative goods. The price fall in turn increases the real income of the buyers of innovative goods, i.e. of the middle generation. Low rates of population growth improve the level of per capita consumption to the middle age groups (“parents”).

Next, we determine the steady state per capita consumption of the grandparents. In the appendix we prove

$$c_{t-1} = \Upsilon \gamma_N^{1-\alpha} k_t^\alpha,$$ (32)

where the auxiliary variable $\Upsilon = \alpha s \Gamma \left( \frac{1 + \gamma_N \gamma_N - s \Gamma}{s\Gamma} \right)^{1-\alpha}$ does not depend on $\gamma_N$. The interesting feature of this equation is the fact that $c_{t-1}$ does not monotonically depend on $\gamma_N$. The first factor $k_t^\alpha$ declines but $\gamma_N^{1-\alpha}$ increases with $\gamma_N$. There are two counteracting effects of $\gamma_N$ on $c_{t-1}$. Firstly, a lower population growth rate increases long run capital intensity. This corresponds to an income effect as it makes higher consumption of all generations possible. Secondly, a lower population growth rate shifts production structure less than the age structure, leaving a gap. Because of this gap, mature good become (relatively) scarcer and more expensive. This is similar to a substitution effect.

It is possible to derive a necessary and sufficient condition for a positive signature of the dependency of $c_{t-1}$ on $\gamma_N$. First, note that the elasticity of $k_t$ with respect to $\gamma_N$ is (equation (25))

$$\frac{\partial k_t}{\partial \gamma_N} \frac{\gamma_N}{k_t} = \frac{-(1+1)\alpha}{1-\alpha}.$$
Therefore
\[
\frac{\partial c_2}{\partial \gamma N} c_2 = -\alpha \frac{(1 + \alpha(1 - \sigma))}{1 - \alpha} + 1 - \alpha
\]
and finally:
\[
\frac{\partial c_2}{\partial \gamma N} c_2 \geq 0 \iff \frac{1 - 3\alpha + \alpha^2}{\alpha^2} \geq (1 - \sigma).
\]
This condition can be visualized by considering the graph of \((1 - 3\alpha + \alpha^2)/\alpha^2\) in the following figure. The function has a zero at 0.38 and increases to infinity for \(\alpha \to 0\). Thus, there is always a solution of \(1 - 3\alpha + \alpha^2 = (1 - \sigma)\) in the interval \((0, 0.38)\). Note that \(\alpha\) is the exponent of capital in the production function. The condition (33) is satisfied in a neighborhood of 0.3, which is plausible value for \(\alpha\). Consider an numerical example. Suppose that \((1 - \sigma) = 0.6\) in this case \(\alpha^* = 0.3496\) holds. For all economies with \(\alpha < \alpha^*\) grandparents would have a lower per capita consumption if \(n\) decreases.

---

We can draw three observation from the condition (33): Whether the per capita consumption of the grandparents depends positively or negatively on \(\gamma N\) merely depends on the production functions (see equations (1), (2) and (8)). The “demand side” represented by the consumption rate \(1 - s\) has no effect on this dependency. Note, that the sign of \(\partial c_{t-1}/\partial \gamma N\) is positive if \(\alpha\) is small. If \(\gamma N\) declines by 1 per cent the ratio \(N_{t-1}/N_t = 1/\gamma N\) increases by one per cent then the ratio \(X_{t-1}/X_t\) increases by \(\alpha\) per cent (see equation

\[\text{It is possible to derive an analytic solution of the critical } \alpha^* \text{ in terms of } \sigma. \text{ It holds}\]
\[
\alpha^* = \frac{1/2(-3 - \sqrt{4(1 - \sigma) + 5})}{1 + (1 - \sigma)}
\]
Therefore, if \( \alpha \) is small the shift of the production structure leaves a gap between the supply of mature goods relative to the number of grandparents. This gap is larger if \( \alpha \) is smaller. Finally, the sign of \( \partial c_{t-1}/\partial \gamma_N \) is positive if \( \sigma \) is large. A large \( \sigma \) implies that mature goods are relatively unproductive in the production of new goods (equation (8)). Thus a shift towards more mature goods dampens productivity in case of a large \( \sigma \).

4 The Model with Endogenous Savings

In this section we check whether a non-neoclassical scenario can prevail if savings are endogenous. So far we assumed an exogenously given rate of saving. However, individuals can adjust their consumption and savings if the growth rate of population changes. It is conceivable that an endogenous saving rate abolishes our non-neoclassical result that the per capita consumption of the old generation can be both positively or negatively related to the rate of population growth. Individuals may adjust their savings in the case of a change of the population growth. This can destroy the non-neoclassical result that pensioners lose in terms of consumption if the population growth rate decreases. Individuals may adjust their savings and mitigate or neutralize the effect found in section 3.

We assume a standard life-time utility function

\[
U_t = u(c_t(t)) + \beta u(c_t(t + 1)), \quad 0 < \beta < 1
\]

with a log felicity function \( u(c) = \log(c) \). Optimality implies the following Euler Equation

\[
\frac{c_t(t + 1)}{c_t(t)} = \beta \text{(rate of return)} = \beta \frac{R_t(t + 1)}{V_t(t)}.
\]

In the steady state the ratio \( \frac{c_{t+1}}{c_t} \) of the life cycle profile of consumption is the same as the ratio \( \frac{c_{t+1}}{c_t} \) of the cross section profile of consumption at a certain point of time. We obtain:

\[
\frac{c_{t-1}}{c_t} = \beta \frac{R_{t-1}}{V_t} = \beta R_{t-1}.
\]

Using equation (31) and (32), the definitions of \( \Upsilon \) and \( \Gamma \) and the fact that \( R_{t-1} = \frac{\sigma \gamma_N}{1 - \alpha} \frac{1 + \gamma_N \gamma_V}{s} - \delta \Gamma \) we deduce the following steady state condition (see the appendix).

\[
\beta \gamma_N \alpha ((1 + \chi)(1 + \gamma_N \gamma_V) + s \sigma (1 - \alpha))^\alpha = \frac{s(\chi(1 + \gamma_N \gamma_V) - s \sigma (1 - \alpha))}{(1 - s)} (34)
\]

In the appendix we also derive second equilibrium condition involving \( s \) and \( \gamma_N \gamma_V \): \n
\[
\gamma_N \gamma_V = \frac{s(1 - \alpha)(1 - \sigma) + \alpha(\chi(1 + \gamma_N \gamma_V))}{s \sigma (1 - \alpha) + \alpha (1 - \chi)(1 + \gamma_N \gamma_V)} (35)
\]

The equation (34) and (35) determine \( s \) and \( \gamma_N \gamma_V \). With latter we can calculate \( R_{t-1} \) and \( \frac{c_{t-1}}{c_t} \). We solved the problem numerically with the following parameters (see the
appendix for the Maple Session): $\alpha = 0.3$, $\sigma = 0.6$, $\beta = 0.8$.

\[
\begin{array}{|c|c|c|c|}
\hline
\gamma_N & 0.9 & 1.0 & 1.1 \\
\hline
R_{t-1} & 9\% & 20\% & 30\% \\
\hline
s & 0.43 & 0.44 & 0.45 \\
\hline
\hat{\varepsilon} & 0.87 & 0.96 & 1.04 \\
\hline
\end{array}
\]

In principle, the non-neoclassical effect found in section 3 could be balanced out if individuals increase their savings in the case of a lower population growth rate. However, for the parameters (and indeed for a broad spectrum of parameters) chosen above we find that individuals decrease their saving rate if the population growth rate declines. Thus with endogenous savings the effect found in section 3 is reinforced. To explain this change in saving note that the rate of return $R_{t-1}$ decreases if population growth rate decreases. This reduces the incentive to save.

5 Conclusion

The vast majority of economies face the problem of an aging population and there is a large literature about the economic consequences of aging. We highlight a so far neglected problem. Our main assumption is that the vintage approach is adequate for consumption as much as for production. According to Becker (1965) consumption resembles production (consumption as home production). Hence, it is merely consistent to apply the vintage approach to consumption. We justify the assumption of a vintage structure of consumption by habit persistence. The home production of households depends upon their technical know-how which varies with their age. Adolescents and young adults acquire the newest technical knowledge more easily and have a stronger incentive to invest in new knowledge than older generations. The former consume new goods and services that their parents are not willing or not able to consume. The latter often maintain throughout their lives the preferences formed in the years of their youth. The thesis that the consumption pattern is shaped in the days of childhood and youth makes the vintage approach attractive for the analysis of consumption.

Our main result is that a declining/aging population is related to a lower steady state consumption of pensioners. This happens if the adjustment of the vintage structure of production to the age structure of the population is unfavorable in terms of the relative productivity of the capital-vintages.

References


Economy, 99. 1142 - 1165.


We suggest that the appendix is published in the internet only.

Appendix

Appendix to Section 2.2 (Temporary Equilibrium)

To derive the equilibrium conditions (24) and (23) we proceed as follows: We can express \( I_t \) in terms of \( L_t \):

\[
I_t = X_t - C_t = K_t^{-\alpha}L_t^{1-\alpha} - (1 - s)(1 - \alpha)K_t^{-\alpha}L_t^{-\alpha}N_1.
\]

Next, we obtain (see equation (9) and (10))

\[
sWN_1 = V_{t+1}K_{t+1} + K_t = V_{t+1}I_t^{1-\sigma} + K_t.
\] (36)

The factor price equations imply

\[
V_{t+1} = \frac{W}{\sigma(1 - \alpha)K_t^{-\alpha}L_t^{-\alpha}I_t^{1-\sigma}}.
\] (37)

If we substitute (37) into (36) we obtain

\[
sWN_1 = \frac{WI_t}{\sigma(1 - \alpha)K_t^{-\alpha}L_t^{-\alpha}I_t^{1-\sigma}} + K_t.
\]

Using equation (36) we obtain

\[
sWN_1 = \frac{W[I_t - (1 - s)(1 - \alpha)N_1]}{\sigma(1 - \alpha)K_t^{-\alpha}L_t^{-\alpha}I_t^{1-\sigma}} + K_t
\]

Hence \( WsN_1(1 - \alpha) = W[LI_t - (1 - s)(1 - \alpha)N_1] + K\sigma(1 - \alpha) \) and finally (24) follows.

The factor price equations imply

\[
\sigma \frac{I_t}{1 - \sigma I_{t-1}} = \frac{\partial K_{t+1}}{\partial I_{t-1}} = \frac{\partial N_{t+1}}{\partial K_{t}} = \left( \frac{K_{t+1}}{K_t} \right)^\alpha.
\]

Using this equation and (37) we obtain (23).

Appendix to Section 2.2 (Proof of the existence of a temporary equilibrium)

The equation for the temporary equilibrium

\[
W(t)
\]

\[
\sigma(1 - \alpha)(1 - \sigma)\left( \frac{K_t(t)}{L_t(t)} \right)\alpha \left( \frac{L_t(t)}{N_1(t)(1 - \sigma)\left( \frac{K_t(t)}{L_t(t)} \right)^\alpha (1 - \sigma)} \right)^\gamma(t + 1)
\]

where

\[
W(t) = \frac{K_t(t)\sigma(1 - \alpha)}{sN_1(t)(1 - \alpha) - L_t(t) + (1 - s)(1 - \alpha)N_1(t)}.
\] (39)
This condition can be rewritten as
\[
\frac{\Upsilon_1 L^{\alpha(1-\sigma)}}{L^\alpha (N - L)^{\alpha(1-\sigma)}} = \frac{\Upsilon_1 L^{1-\sigma}}{(N - L)^{\alpha(1-\sigma)}} = \frac{\Upsilon_2}{\Upsilon_3 N - L}
\]
or
\[
\Upsilon_4 L^{1-\sigma} = \frac{(N - L)^{\alpha(1-\sigma)}}{\Upsilon_3 N - L}
\]
The right hand side has a pole at $\Upsilon_3 N$. The left hand side is a concave function satisfying the Inada-condition.

**Appendix to 2.3 (Steady state conditions)**

We will need the following relations:

\[
\frac{R_t}{R_{t-1}} = \frac{K_{t-1} L_t}{K_{t-1} L_{t-1}}
\]
\[
\frac{1}{R_{t-1}} = \frac{1}{P_{t-1} \alpha K_{t-1}^{\alpha-1} L_{t-1}^{1-\alpha}} = \frac{(1 - \alpha) K_{t-1}}{\alpha W L_{t-1}}
\]

and

\[
V_{t+1} K_{t+1} + V_t K_t = V_{t+1} \gamma_N K_t + V_t K_t = (\gamma_N \gamma_V + 1) K_t = s W N_1.
\]

This gives the following steady state condition

\[
\frac{N_1}{K_t} = \frac{1 + \gamma_N \gamma_V}{s W}.
\] (40)

We define $\chi = (s \sigma_1 (1 - \alpha) + (1 - s)(1 - \alpha)) N_1(t)$. With this notation the temporary equilibrium condition (24) can be rewritten as

\[
W = \left( \frac{\chi (1 + \gamma_N \gamma_V)}{s} - \sigma_1 (1 - \alpha) \right) k_t = \Gamma k_t
\] (41)

Thus

\[
\frac{N_1}{K_t} = \frac{1 + \gamma_N \gamma_V}{s \Gamma k_t}
\] (42)
We deduce (28):

\[
\gamma_V = \frac{R_t + 1}{R_{t-1}}
\]

\[
= \frac{R_t}{R_{t-1}} + \frac{1}{R_{t-1}}
\]

\[
= \frac{1}{k} \frac{K_{t-1}}{N_1 - L_t} + \frac{(1 - \alpha)}{\alpha W} \frac{K_{t-1}}{N_1 - L_t}
\]

\[
= \left( \frac{1}{k} + \frac{(1 - \alpha)}{\alpha W} \right) \frac{K_{t-1}}{(N_1 - L_t)}
\]

\[
= \left( \frac{1}{k} + \frac{(1 - \alpha)}{\alpha kT} \right) \left( \frac{(N_1 - L_t)\gamma_N}{k} \right)^{-1}
\]

\[
= \left( \frac{1}{k} + \frac{(1 - \alpha)}{\alpha k} \right) \left( \gamma_N \frac{1 + \gamma_N \gamma_V}{sTk} - \gamma_N \right)^{-1}
\]

\[
= 1 + \frac{1 - \alpha}{\alpha T} \gamma_N \frac{1 + \gamma_N \gamma_V}{sT} - 1)
\]

\[
= \frac{s(1 + \alpha T - \alpha)}{\alpha \gamma_N (1 + \gamma_N \gamma_V - sT)}
\]

The following is useful

\[
\frac{N_{t-1}}{K_{t-1}} = \frac{1 + \gamma_N \gamma_V}{sW} = \frac{(1 + \gamma_N \gamma_V) L_t}{sT K_t}
\]

which implies

\[
\frac{N_t}{L_t} = 1 + \gamma_V \gamma_N.
\]

Note

\[
\gamma_V = V_{t+1} = \frac{W}{\sigma_1(1 - \alpha)K_t L_t \alpha T_{t-1}^{-1} T_{t-1}^{-1}}
\]

Now we can derive (25):

\[
\gamma_V = k_1 \Gamma \left( \frac{N_1 L_1}{L_t} \right) \frac{a(1-\sigma)}{\sigma(1-\alpha)(1-\sigma) \gamma_N a(1-\sigma) k_t^a}
\]

\[
= k_1 \Gamma \left( \frac{N_1 L_1}{L_t} - 1 \right) \frac{a(1-\sigma)}{\sigma(1-\alpha)(1-\sigma) \gamma_N a(1-\sigma) k_t^a}
\]

\[
= k_1 \Gamma \left( \frac{N_1 L_1}{L_t} - 1 \right)^{a(1-\sigma)}
\]

\[
= k_1^{1-\alpha} \Gamma \left( \frac{(1 + \gamma_N \gamma_V)}{sW} k_t - 1 \right)^{a(1-\sigma)}
\]

\[
= k_1^{1-\alpha} \Gamma \left( \frac{(1 + \gamma_N \gamma_V)}{sW} k_t - 1 \right)^{a(1-\sigma)}
\]

\[
= \frac{k_1^{1-\alpha} \Gamma \left( \frac{(1 + \gamma_N \gamma_V)}{sW} k_t - 1 \right)^{a(1-\sigma)}}{\sigma(1-\alpha)(1-\sigma) \gamma_N a(1-\sigma)}
\]

\[
= k_1^{1-\alpha} \Gamma \left( \frac{(1 + \gamma_N \gamma_V)}{sW} k_t - 1 \right)^{a(1-\sigma)}
\]

\[
= \frac{k_1^{1-\alpha} \Gamma \left( \frac{(1 + \gamma_N \gamma_V)}{sW} k_t - 1 \right)^{a(1-\sigma)}}{\sigma(1-\alpha)(1-\sigma) \gamma_N a(1-\sigma)}
\]

16
Appendix to section 3 (Proof of equation (32))

\[ P_{t-1}C_{t-1} = R_tK_t + V_tK_t + R_{t-1}K_{t-1} \]
\[ \Rightarrow \frac{P_{t-1}}{R_{t-1}}C_{t-1} = \frac{R_t + V_t}{R_{t-1}}K_t + K_{t-1} \]
\[ \Rightarrow \frac{P_{t-1}}{R_{t-1}}C_{t-1} = (\gamma_V + \frac{1}{\gamma_N})K_t \]
\[ \Rightarrow C_{t-1} = (\gamma_V + \frac{1}{\gamma_N})\frac{P_{t-1}}{R_{t-1}}K_t \]
\[ \Rightarrow C_{t-1} = \alpha \left( \gamma_V + \frac{1}{\gamma_N} \right) \left( \frac{K_t}{\gamma_N L_t \frac{L_t}{L_{t-1}}} \right)^{\alpha-1} K_t \]
\[ \Rightarrow C_{t-1} = \alpha \left( \gamma_V + \frac{1}{\gamma_N} \right) \left( \frac{k_t}{\gamma_N \frac{L_t}{L_{t-1}}} \right)^{\alpha-1} K_t \]

Note

\[ \frac{L_t}{L_{t-1}} = \left( \frac{N_t}{L_t} - 1 \right)^{-1} = \left( \frac{1 + \gamma_N \gamma_V}{s\Gamma} - 1 \right)^{-1} = \left( \frac{s\Gamma}{1 + \gamma_N \gamma_V - s\Gamma} \right) \]

We obtain

\[ c_{t-1} = \frac{C_{t-1}}{N^2} = \alpha \left( \gamma + \frac{1}{\gamma_N} \right) \left( \frac{k_t}{\gamma_N} \left( \frac{s\Gamma}{1 + \gamma_N \gamma_V - s\Gamma} \right) \right)^{\alpha-1} K_t \]
\[ \frac{1}{N_1(t)} \]
\[ \gamma_N \]
\[ \gamma_N \]
\[ \gamma_N \]
\[ \gamma_N \]
\[ \gamma_N \]
\[ \gamma_N \]
\[ \gamma_N \]
\[ \gamma_N \]
\[ \gamma_N \]
\[ \gamma_N \]
\[ \gamma_N \]
\[ \gamma_N \]
\[ \gamma_N \]
\[ \gamma_N \]
\[ \gamma_N \]
\[ \gamma_N \]

or

\[ c_{t-1} = \Upsilon k_t^\alpha n^{1-\alpha}, \]

Appendix to section 3 (Proof of Condition (33))

\[ \frac{\partial c_{t-1}}{\partial n} \leq 0 \]
\[ -(1 - \sigma)\alpha^2 - \alpha + (1 - \alpha)^2 \geq 0 \]
\[ \frac{(1 - \alpha)^2 - \alpha}{\alpha^2} \leq (1 - \sigma) \]
\[ \frac{1 - 3\alpha + \alpha^2}{\alpha^2} \leq (1 - \sigma) \]

17
Appendix to section 4

\[ R_{t-1} = \frac{\alpha W L_{t-1}}{(1-\alpha)K_{t-1}} = \frac{\alpha W}{(1-\alpha)} \frac{L_{t-1} N_1}{K_{t-1}} \]
\[ = \frac{\alpha W}{(1-\alpha)} \frac{L_{t-1} N_1}{K_{t-1}} \gamma_N \]
\[ = \frac{\alpha W}{(1-\alpha)} \frac{1 + \gamma_N \gamma_V - s\Gamma}{s\Gamma k_t} \]
\[ = \frac{\alpha W}{(1-\alpha)} \frac{1 + \gamma_N \gamma_V - s\Gamma}{s\Gamma k_t} \gamma_N \]
\[ = \frac{\alpha W}{(1-\alpha)} \frac{1 + \gamma_N \gamma_V - s\Gamma}{s} \]

Using equations (31) and (32) we obtain

\[ \beta R_{t-1} = \frac{c_{t-1}}{c_t} \gamma_N^{1-\alpha} \]

Because of \( R_{t-1} = \frac{\alpha \gamma_N}{(1-\alpha)^{(1+\gamma_N \gamma_V - s\Gamma)}} \) we obtain

\[ \frac{\alpha \beta \gamma_N}{(1-\alpha)} \frac{1 + \gamma_N \gamma_V - s\Gamma}{s} = \gamma_N^{1-\alpha} \]
\[ \Rightarrow \alpha \beta \gamma_N^{\alpha} (1 + \gamma_N \gamma_V - s\Gamma) = \gamma_N^{1-\alpha} \]

Using the definition of \( \mathcal{T} \) we derive

\[ \beta \gamma_N^{\alpha} (1 + \gamma_N \gamma_V - s\Gamma)^\alpha = \frac{s(\Gamma)^\alpha}{(1-s)} \]

This is the equation (34).

Note that (see equation (28))

\[ \gamma_N \gamma_N = \frac{s(1-\alpha) + \alpha s\Gamma}{\alpha(1 + \gamma_N \gamma_V - s\Gamma)} \]
\[ = \frac{s(1-\alpha) + \alpha(\chi + (1 + \gamma_N \gamma_V) - s\sigma(1-\alpha))}{\alpha(1 + \gamma_N \gamma_V - \chi(1 + \gamma_N \gamma_V) + s\sigma(1-\alpha))} \]
\[ = \frac{s(1-\alpha) + \alpha(\chi + (1 + \gamma_N \gamma_V) - s\sigma(1-\alpha))}{\alpha((1-\chi)(1 + \gamma_N \gamma_V) + s\sigma(1-\alpha))} \]
\[ = \frac{s(1-\alpha)(1-\sigma) + \alpha(\chi(1 + \gamma_N \gamma_V))}{s\sigma(1-\alpha)\alpha + \alpha(1-\chi)(1 + \gamma_N \gamma_V)} \]

Hence, we have proved (35).
Maple Sessions

We will use the equation (35) in the following maple session.

```maple
given expressions
```

Next, we use this expression for \( s \) and the equation (34) to calculate \( s, \hat{c} \) and \( R_{t-1} \).

```maple
\( s := \hat{c} \\); 
\( R := \alpha y n ((1 + x - (1 - \alpha) (1 - s (1 - \sigma))) (1 + x) + s \sigma (1 - \alpha)) / (1 - \alpha) s \)
```

19
\[
\begin{align*}
R &= -yn(1 + x - (1 - \alpha))(1 + \frac{\alpha(1 + x)(\alpha x + \alpha - 1)(1 - \sigma)}{(\alpha - 1)\%1})(1 + x) \\
&= \frac{\alpha(1 + x)(\alpha x + \alpha - 1)\sigma(1 - \alpha)}{(\alpha - 1)\%1}(\alpha - 1)\%1/((1 - \alpha)(1 + x)(\alpha x + \alpha - 1)) \frac{\alpha x^2 - 2\alpha x - \alpha x^2 + \alpha x - \alpha + 1 - \sigma}{\alpha x^2 - 2\alpha x - \alpha x^2 + \alpha x - \alpha + 1 - \sigma}
\end{align*}
\]

\[
\text{expression}:=\text{subs}(\text{sigma}=0.6, \text{yn}=0.9, \text{beta}=0.8, \text{s}=\text{f}(x), \text{alpha}=0.3, (\text{beta*(1-s)})^*\frac{(1/\alpha)\text{yn}*((1-(1-\alpha)*(1-s*(1-sigma)))*(1+x)+s*\text{sigma}*(1-\alpha))}{s^*(1/\alpha)*(1-s*(1-sigma))});
\]

\[
\text{x1}:=\text{fsolve}(\text{expression}, x, 0..\text{infinity});
\]

\[
\text{s1}:=\text{subs}(\text{sigma}=0.6, \text{yn}=0.9, \text{beta}=0.8, \text{s}=\text{f}(x), \text{alpha}=0.3, \text{x=x1}, \text{s});
\]

\[
\text{chat}:=0.8*\text{subs}(\text{s}=\text{s1}, \text{x}=\text{x1}, \text{alpha}=0.3, \text{sigma}=0.6, \text{yn}=0.9, \text{R});
\]

\[
\text{R1}:=\text{subs}(\text{s}=\text{s1}, \text{x}=\text{x1}, \text{alpha}=0.3, \text{sigma}=0.6, \text{yn}=0.9, \text{R});
\]

\[
\text{x1} := 1.496443103
\]

\[
\text{s1} := 0.4352007416
\]

\[
\text{chat} := .8763104024
\]

\[
\text{R1} := 1.095388003
\]

\[
\text{x1} := 1.486394468
\]

\[
\text{s1} := 0.4442803029
\]

\[
\text{chat} := .9583291808
\]

\[
\text{R1} := 1.197911476
\]

\[
\text{x1} := 1.477422649
\]

\[
\text{s1} := 0.4525342042
\]

\[
\text{chat} := 1.039422442
\]

\[
\text{R1} := 1.299278052
\]